Computers in Civil Engineering 53:081 Spring 2000

Lecture #14

Improvements to Gauss Elimination

Improvements to Gauss Elimination Method

- **Use of increased precision**
  - **Reduces round-off errors**
- **Pivoting (partial)**
  - **Eliminates division by small numbers**
- **Scaling**
  - **Reduces round-off errors**

Partial Pivoting (Switching Equations)

$$
\begin{aligned}
2x_2+3x_3 &= 8 \\
4x_1-6x_2+7x_3 &= -3 \\
2x_1-x_2+6x_3 &= 5
\end{aligned}
$$

Select the row with largest coefficient for pivot

$$
\begin{aligned}
4x_1-6x_2+7x_3 &= -3 \\
2x_1-x_2+6x_3 &= 5 \\
2x_2+3x_3 &= 8
\end{aligned}
$$

Scaling

Consider the following system of linear equations:

$$2x_1+100{,}000x_2 = 100{,}000$$
$$x_1+x_2 = 2$$

The correct solution is: $x_1 = 1.00002$ and $x_2=0.99998$

Keeping 3 significant digits: $x_1=1.00$ and $x_2=1.00$

Solving using naive Gauss Elimination:

$$2x_1+100{,}000x_2 = 100{,}000$$
$$-50{,}000x_2 = -50{,}000$$

Produces (wrong solution) $x_1 = 0.00$ and $x_2 = 1.00$

Use of Scaling

Original system:

$$2x_1 + 100{,}000x_2 = 100{,}000$$
$$x_1 + x_2 = 2$$

Scale:

$$0.00002x_1 + x_2 = 1$$
$$x_1 + x_2 = 2$$

Select the pivot:

$$x_1 + x_2 = 2$$
$$0.00002x_1 + x_2 = 1$$

Eliminate x_1:

$$x_1 + x_2 = 2$$
$$x_2 = 1.00$$

Solution:

$$x_1 = x_2 = 1$$

Solving
Systems of Linear Equations

Gauss-Jordan Method

Gauss-Jordan Elimination Method

- Eliminates variables from all equations at once
- Equations are normalized
- Results in a diagonal matrix (no need for back-substitution)

Forward Elimination: Reduces Equations Identity Matrix

$$\begin{matrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{matrix} \quad \Rightarrow \quad \begin{matrix} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{matrix}$$

Solution

Steps of Gauss-Jordan Method

- Step 1. Form an augmented matrix $n\times(n+1)$
- Step 2. Normalize the first row (divide by the pivot)
- Step 3. Eliminate x_1 by subtracting row 2-row 1 multiplied by a_{21}
- Step 4. Eliminate other variables repeating steps 2 & 3
- Step 5. Find the solution in the right-hand-side vector

Example

Consider the system of linear equations:

$$\begin{aligned} 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10.0x_3 &= 71.4 \end{aligned}$$

Augmented matrix form:

Pivot → 3

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10.0 & 71.4 \end{array}\right]$$

Example (continued ...)

Normalize--divide by the pivot (3)

1	-0.0333333	-0.0666667	2.61667
0.1	7	-0.3	-19.3
0.3	-0.2	10	71.4

Eliminate x_1:

1	-0.0333333	-0.0666667	2.61667
0	7.00333	-0.293333	-19.5617
0	-0.190000	10.0200	70.6150

Example (continued ...)

Normalize the second row--divide by 7.00333

1	-0.0333333	-0.0666667	2.61667
0	1	-0.0418848	-2.79320
0	-0.190000	10.0200	70.6150

Eliminate x_2 from rows 3 and 1 ← This is different than Gauss elimination

1	0	-0.0680629	2.52356
0	1	-0.0418848	-2.79320
0	0	10.0120	70.0843

Example (continued ...)

Normalize the third row--divide by 10.0120

1	0	-0.0680629	¦	2.52356
0	1	-0.0418848	¦	-2.79320
0	0	1	¦	7.00003

Eliminate x_3 from rows 1 and 2

1	0	0	¦	3.00000
0	1	0	¦	-2.50001
0	0	1	¦	7.00003

Pseudocode for Gauss-Jordan

```
do k=1,n                                 !for each equation
  pivot=a(k,k)                           !select pivot
  do j=1,n+1                             !for each column
    a(k,j)=a(k,j)/pivot                  !normalize
  enddo
  do i=1,n                               !for each equation
    if(i /= k)                           !skip the pivot eq
      aux=a(i,k)                         !select multiplier
      do j=1,n+1                         !for each column
        a(i,j)=a(i,j)-aux*a(k,j)
      enddo                              !eliminate x's
    endif
  enddo
enddo
```

Pitfalls of the Gauss-Jordan Elimination Method

- Similar to Gauss Elimination
- Division by zero pivot
- Round-off errors
- Slower than Gauss Elimination

Main Advantage:
Simple way to invert matrices

Matrix Inversion

$$[A][A]^{-1}=[A]^{-1}[A]=[I]$$

System of Linear Equations

$$[A]\{X\}=\{C\} \Rightarrow \{X\}=[A]^{-1}\{C\}$$

Matrix Inversion

$$[A][X]=[C]$$

If $[C]=[I]$ then $[X]=[A]^{-1}$

Matrix Inversion
Gauss-Jordan Elimination Method

Step 1. Form augmented matrix with identity matrix:

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

The coefficients of A

Step 2. Solve using Gauss-Jordan.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right]$$

This part is the inverse of A

Next: HYDRAULICS EXAMPLE